Monodromy of Compositions of Toroidal Belyi Maps

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Abstract

Say that $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Dynamical Belyĭ map. Given any Toroidal Belyĭ map $\gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, the composition $\beta \circ \gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is also a Toroidal Belyĭ map. There is a group $\mathrm{Mon}(\beta)$, the monodromy group, which contains information about the symmetries of a Belyĭ ı map β. It is well-known that, for any Toroidal Belyĭ ı map γ, (i) there is always a surjective group homomorphism $\mathrm{Mon}(\beta \circ \gamma) \twoheadrightarrow \mathrm{Mon}(\beta)$, and (ii) the monodromy group $\mathrm{Mon}(\beta \circ \gamma)$ is contained in the $\mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$.

In this project, we study how the three groups $\mathrm{Mon}(\beta)$, $\mathrm{Mon}(\beta \circ \gamma)$, and $\mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$ compare as we vary over Dynamical Belyĭ maps β. This is work done as part of the Pomona Research in Mathematics Experience (NSA H98230-21-1-0015).

1 Background

1.1 Elliptic Curves

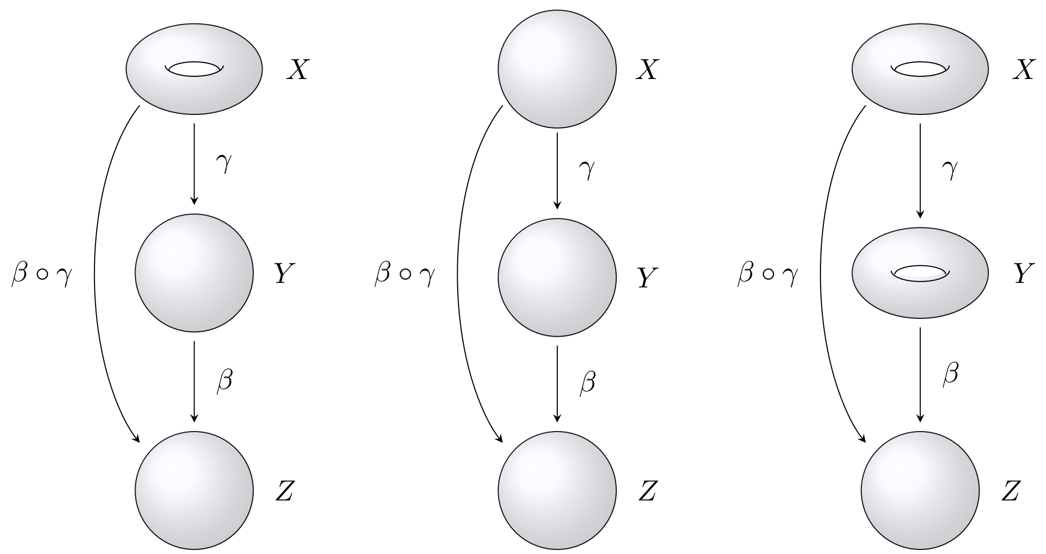
An **elliptic curve** E is the set of all points (x, y) satisfying a nonsingular equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for complex a_1, a_2, a_3, a_4, a_6.
Every elliptic curve E has an abelian group structure isomorphic to that of $\mathbb{C}/\Lambda$ where

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$$

for $\mathbb{R}$-linearly independent $\omega_1, \omega_2 \in \mathbb{C}$. Geometrically, every elliptic curve E is a torus $T^2(\mathbb{R})$.



Our Case Wood's Case Zeytin's Case

1.2 Toroidal Belyĭ Maps

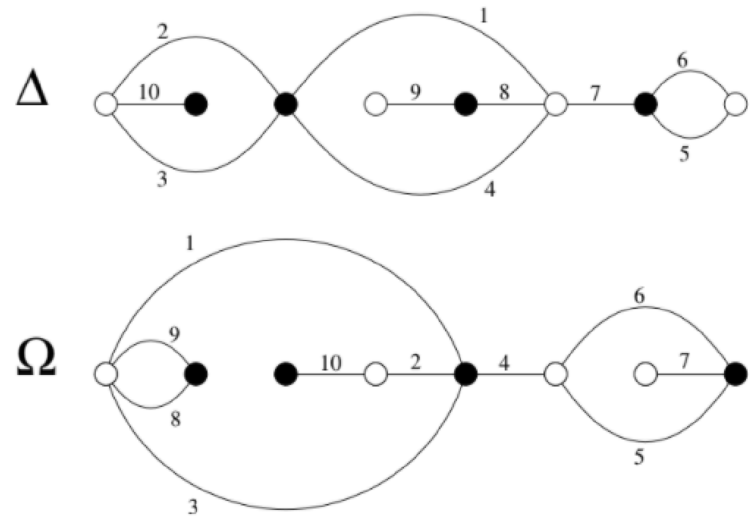
Let X be some Riemann surface. A **Belyĭ map** $\gamma : X \to \mathbb{P}^1(\mathbb{C})$ can be defined as a mapping of a Riemann surface to a Riemann sphere with at most three branch points, which we take to be $\{0, 1, \infty\}$. A **Belyĭ pair** (X, γ), is composed of the Riemann surface and its corresponding Belyĭ map. If we fix E to be an elliptic curve as defined above, we can define the map $\gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Such a map from an elliptic curve to a Riemann sphere is known as a **Toroidal Belyĭ map**. As such, (E, γ) is known as a toroidal Belyĭ pair. Given a projective point $\omega = \omega_1/\omega_0 \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, we can consider the inverse image of such a Belyĭ map:

$$\gamma^{-1}(\omega) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; \begin{array}{r} (y^2 + a_1xy + a_3y) - (x^3 + a_2x^2 + a_4x + a_6) = 0 \\ \omega_0 p(x,y) - \omega_1 q(x,y) = 0 \end{array} \right\}$$

The degree of a Belyĭ map is $\deg \gamma = |\gamma^{-1}(\omega)|$ whenever ω is not a critical value.

1.3 Dessin d'Enfants

Given a Belyĭ pair (E, γ) we define the sets $B = \gamma^{-1}(\{0\})$ and $W = \gamma^{-1}(\{1\})$. We refer to B as the set of black vertices and W as the set of white vertices. The bipartite graph embedded in E with vertices B, W and edges $\gamma^{-1}([0, 1])$ is called a **Dessin d'Enfant**.
The degree of a Belyĭ map γ is equal to the number of edges in its dessin d'enfant.



1.4 Monodromy Groups

Consider the multiset:

$$\mathcal{D} = \{\{e_P \mid P \in B\}, \{e_P \mid P \in W\}, \{e_P \mid P \in F\}\}$$

of three partitions of N for some indexing sets B, W, and F such that $N = |B| + |W| + |F|$. Then $\mathcal{D}$ is the degree sequence for some toroidal Belyi pair (E, γ) with $\deg \gamma = N$ if and only if there exist permutations $\sigma_0, \sigma_1, \sigma_\infty \in S_N$ such that we have the following three properties:

- Each of the permutations in $\mathcal{D}$ is a product of disjoint cycles with corresponding cycle types.
- G is a transitive subgroup of S_N
- $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$

A group of the form $G = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ that satisfies these properties is said to be a **monodromy group**. For example, consider the degree sequence $\mathcal{D} = \{\{1,4\}, \{1,4\}, \{5\}\}$ for $N = 5$. Some possible monodromy groups include the following:

$$\begin{array}{ll}
\sigma_0 = (2)(1354) & \sigma_0 = (3)(1254) \\
\sigma_1 = (4)(1352) & \sigma_1 = (5)(1243) \\
\sigma_\infty = (12345) & \sigma_\infty = (12345) \\
\Rightarrow \quad G \simeq S_5 & \Rightarrow G \simeq F_{20} \simeq Z_5 \rtimes Z_4
\end{array}$$

This illustrates the fact that the degree sequence for a particular monodromy group is not necessarily unique. In particular, there are at least two Belyĭ pairs (S, β) associated with this degree sequence.

1.5 Wreath Products

Given any two groups N, H and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$ we can construct the **semidirect product** $N \rtimes H$ as follows:

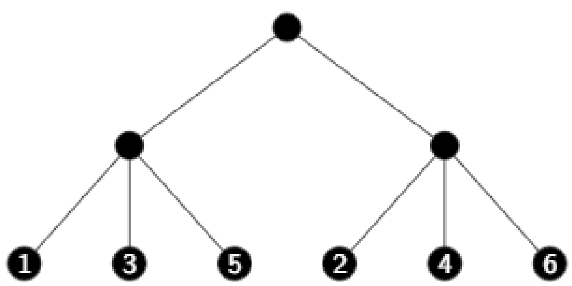
- The underlying set is the product $N \times H$.
- The binary operation $\star$ is defined as

$$(n_1, h_1) \star (n_2, h_2) = (n_1\varphi(h_1)n_2, h_1h_2).$$

Let G be a group and $H \leq S_n$ for some non-negative integer n. Then we can form the **wreath product** as

$$G \wr H = G^n \rtimes H$$

where H acts on G^n by permuting the n copies of G.



Example. If we imagine that each of the sets of 3 labelled vertices in the diagram to the left is a copy of G, then we can say that we have $n = 2$ copies of G. Imagine that the diagram functions like a mobile such that it can be rotated, effectively swapping the vertices around. We can either swap the "arms", or the two vertices in the middle of the diagram, or we can rotate the 3 "hands" on each arm so long as their adjacencies are unchanged (i.e. 1 is to the left of 3 which is to the left of 5 which is to the left of 1.) We can even do some combination of the two, choosing how many times (if any) to rotate each set of hands, and whether or not to swap the arms.

If we read the bottom vertices across from left to right, we start with 135 246. But if we rotate our mobile, we can end with 513 624, or 462 135, etc. In fact, there are 18 possible arrangements of the vertices as there are possible arrangements for each set of hands and 2 possible arm arrangements, yielding $3 * 3 * 2 = 18$.

Now, if we return to our notation for the wreath product, we have that G is a set of 3 hands, so $|G| = 3$ and there are $n = 2$ copies of G, meanwhile H is the set of arms, so $|H| = 2$. For the wreath product $G \wr H$, we have that $|G \wr H| = |G|^n |H|$, and by substituting we have that $3^2 * 2 = 18$. Therefore, the size of the wreath product is equal to the number of possible arrangements (or permutations) as H acts on n copies of G.

Given Belyǐ maps β and γ we have the following group homomorphism:

$$\begin{aligned} \mathrm{Mon}(\beta\gamma) &\to \mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\gamma) \\ \rho_{\beta\gamma}(\lambda) &\mapsto (\rho_{\gamma^*}(f_\lambda), \rho_\beta(\lambda)) \,. \end{aligned}$$

Here $\rho_\beta(\lambda)$ denotes the monodromy representation of λ.

2 Tools

2.1 Jacob Bond's Thesis

From Jacob Bond's thesis, of note were two theorems and the idea of an extending pattern. These concepts are highlighted below.

Corollary (p.71). The monodromy group $\mathrm{Mon}(\beta\gamma)$ of the composition of a dynamical Belyĭ map β and a Belyĭ map γ is isomorphic to a subgroup of the wreath product $\mathrm{Mon}(\gamma) \wr_{E_\beta} \mathrm{Mon}(\beta)$. Moreover, this isomorphism is given by

$$\begin{aligned} \mathrm{Mon}(\beta\gamma) &\to \varphi_\gamma(\pi_1^z) \leq \mathrm{Mon}(\gamma) \wr_{E_\beta} \mathrm{Mon}(\beta) \\ \rho_{\beta\gamma}(\lambda) &\mapsto (\rho_{\gamma^*}(f_\lambda), \rho_\beta(\lambda)). \end{aligned}$$

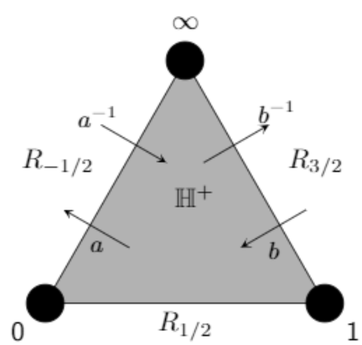
Theorem 4.18 (p.76). Let β be a dynamical Belyĭ map with constellation (τ_0, τ_1) and extending pattern (f_0, f_1). Let φ denote the homomorphism

$$\begin{aligned} g_0 &\mapsto (f_0, \tau_0) \\ g_1 &\mapsto (f_1, \tau_1). \end{aligned}$$

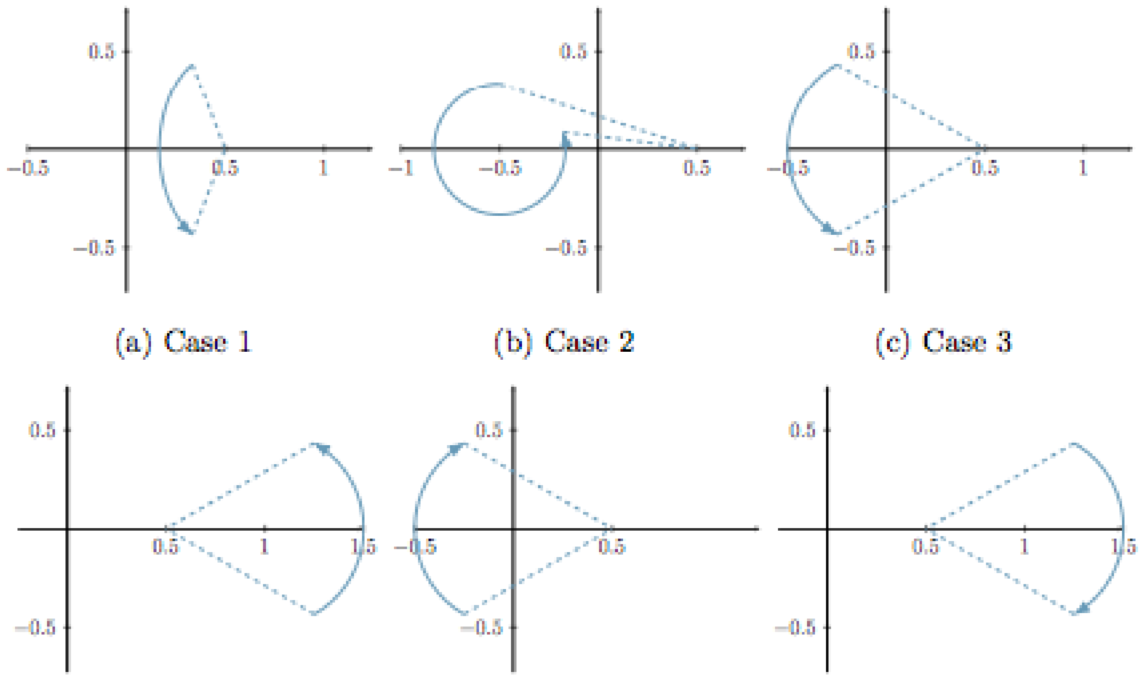
Define $A := \varphi(\ker \rho_\beta)$. Then for any Belyĭ map γ

$$\mathrm{Mon}(\beta\gamma) \cong \rho_{\gamma^*}(A) \rtimes \mathrm{Mon}(\beta).$$

Extending Patterns. The extending pattern is a pair of functions (f_0, f_1) for a map β determined by it's dessin d'enfant. In order to compute the extending pattern, we must follow the counterclockwise cycles of edges around the vertices of the dessin while utilizing the following 6 rules. Each edge is assigned a value, $1, a, b, a^{-1}, b{-}1$ according to the following rules. The diagram below showcases a simplified version of these rules for a triangular dessin.



1. If $p \subseteq \mathcal{R}_{1/2}$, then $p^{\circlearrowleft} \simeq_p 1$.
2. If either $p(0), p(1) \in \overline{\mathbb{H}^+}$ or $p(0), p(1) \in \mathbb{H}^-$ and either $p \subseteq \mathcal{R}_{-1/2}$ or $p \subseteq \mathcal{R}_{3/2}$, then $p \simeq {}_p 1$
3. If $p(0) \in \overline{\mathbb{H}^+}, p(1) \in \mathbb{H}^-$, and $p \subseteq \mathcal{R}_{-1/2}$, then $p^{\circlearrowleft} \simeq_p a$.
4. If $p(0) \in \mathbb{H}^-, p(1) \in \overline{\mathbb{H}^+}$, and $p \subseteq \mathcal{R}_{3/2}$, then $p^{\circlearrowleft} \simeq_p b$.
5. If $p(0) \in \mathbb{H}^-, p(1) \in \overline{\mathbb{H}^+}$, and $p \subseteq \mathcal{R}_{-1/2}$, then $p^{\circlearrowleft} \simeq_p a^{-1}$.
6. If $p(0) \in \overline{\mathbb{H}^+}, p(1) \in \mathbb{H}^-$, and $p \subseteq \mathcal{R}_{3/2}$, then $p^{\circlearrowleft} \simeq_p b^{-1}$.



(a) Case 1 (b) Case 2 (c) Case 3

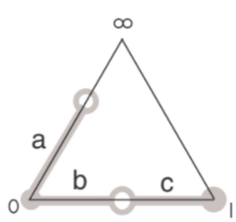
(d) Case 4 (e) Case 5 (f) Case 6

2.2 Melanie Wood's Paper

Melanie Wood uses the composition $\beta \circ \gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, mapping a sphere to a sphere to a sphere.

Consider the following example:

Example 3.7 (p.732). $\gamma(t) = -27(t^3 - t^2)/4$ The extending pattern of γ is shown in the figure below.



$[x_\gamma, y_\gamma] = [(ab), (b, c)]$. The edges of Γ is denoted $1, 2, ..., d$.

$x_{\gamma(\Gamma)}$ is composed of all the cycles of the form

$$((k, a)(x_\Gamma k, b)(x_\Gamma k, a)(x_\Gamma^2 k, b)(x_\Gamma^2 k, a)...)$$

and

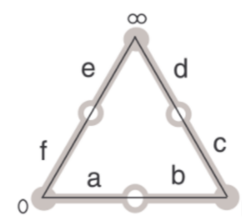
$$((k, c)(y_\Gamma k, c)(y_\Gamma k, c)...)$$

Consider another example:

Example 3.8 (p.733). $\xi(t) = 27t^2/(4(t^2 - t + 1)^3)$.

The extending pattern of ξ is shown in the figure below.

$[x_\xi, y_\xi] = [(af)(bc)(de), (ab)(cd)(ef)]$. The edges of Γ is denoted $1, 2, ..., d$ and $z_\Gamma = (x_\Gamma y_\Gamma)^{-1}$.



$x_{\xi(\Gamma)}$ is composed of all the cycles of the forms

$$((k,a)(k,f)(x_\Gamma k,a)(x_\Gamma k,f)(x_\Gamma^2,a)(x_\Gamma^2,f)...),$$

$$((k,c)(k,b)(y_\Gamma k,c)(y_\Gamma k,b)(y_\Gamma^2,c)(y_\Gamma^2,b)...)$$

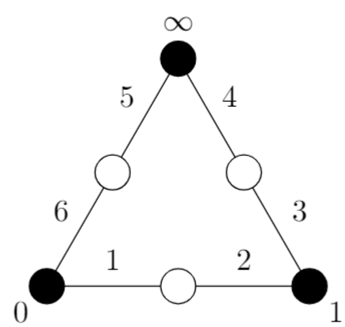
and for $1 \leq k \leq d$:

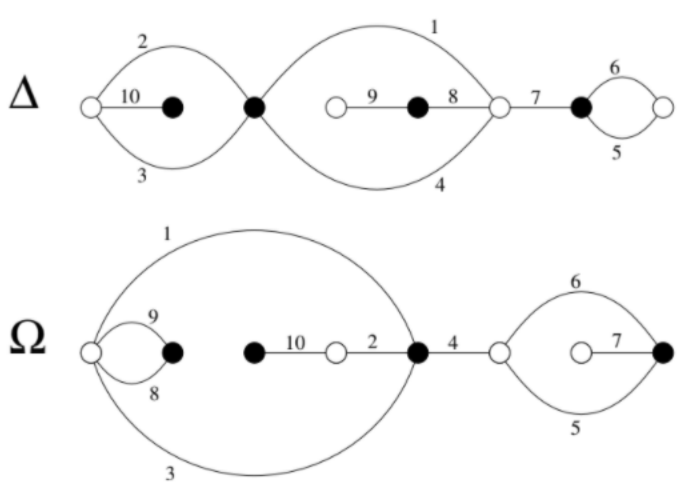
$$((k,e)(k,d)(z_\Gamma k,e)(z_\Gamma k,d)(z_\Gamma^2,e)(z_\Gamma^2,d)...)$$

$y_{\xi(\Gamma)}$ is composed of all the cycles of the form $((k,a)(k,b)), ((k,c)(k,d))$, and $((k,e)(k,f))$, for $1 \leq k \leq d$.

In the notation from Jacob Bond's thesis, for $\gamma = \Delta, \Omega$ and $\beta = \xi$, and if we relabel the edges numerically for the sake of clarity (as shown below), then we have

$$\tau_0 = (1,6)(2,3)(4,5)$$
$$\tau_1 = (1,2)(3,4)(5,6)$$
$$f_0 = [1, b, 1, b^{-1}a^{-1}, 1, a]$$
$$f_1 = [1,1,1,1,1,1]$$





Proposition 3.9. Let Δ be the dessin corresponding to the permutation pair

$[(1234)(567)(89), (1837)(2310)(56)]$

and Ω be the dessin corresponding to the permutation pair

$[(1234)(567)(89), (1389)(210)(456)]$

Then Δ and Ω have the same valency lists, automorphism groups, monodromy groups, cartographic groups, and rational Nielsen classes. However, the M_ξ groups of Δ and Ω differ in size. Thus Δ and Ω are in different $G_\mathbb{Q}$-orbits.

2.3 Belyĭ Lattès Maps by Ayberk Zeytin

Let E be an elliptic curve given by $E : y^2 = x^3 + 1$. Consider the toroidal Belyĭ map

$$\phi : E \to \mathbb{P}^1$$

given by

$$\phi : P = (x, y) \mapsto z = \frac{1-y}{2}.$$

For any positive integer N, the multiplication by N map on E, $[N]$ yields a dynamical Belyĭ map $B_N : \mathbb{P}^1 \to \mathbb{P}^1$ given by $B_N(\phi(P)) = \phi([N])$. Then, B_N has degree N^2 and the B_N are called **Lattès maps**.

A few cases are shown in the table below. We will focus on the $n = 2$ and $n = 3$ case, but the $n = 4$ case is also shown to illustrate another example.

n	B_n	$\mathrm{Mon}(B_n)$	$\mathrm{Mon}(B_n \circ \phi)$
2	$\frac{(z-1)(z+1)^3}{8(z-1/2)^3}$	A_4	A_4
3	$\frac{(z^3+3z^2-6z+1)^3}{27z(z-1)(z^2-z+1)^3}$	He_3 (Heisenberg of order 27)	He_3
4	$\frac{z(z^5+8z^4-32z^3+28z^2-10z+4)^3}{(4z^5-10z^4+28z^3-32z^2+8z+1)^3}$	$(C_4 \times C_4) \rtimes C_3$	$(C_4 \times C_4) \rtimes C_3$

The composition $\beta : P = (x, y) \mapsto B_N(\phi(P)) = \phi([N]P)$ is a Toroidal Belyi map of degree $3 \cdot N^2$. When n=2, the degree of B_n is $3 \cdot 2^2 = 12$. When n=3, the degree of B_n is $3 \cdot 3^2 = 27$. Likewise, when n=4, the degree of B_n is $3 \cdot 4^2 = 48$. Note that the degree of $\mathrm{Mon}(B_n \circ \phi)$ is equal to the degree of $\mathrm{Mon}(B_n)$ in every case.

For the case of n=2, the extending pattern is as follows.

$$\tau_0 = (1, 3, 4) \quad f_0 = [1, b, a, a^{-1}]$$
$$\tau_1 = (2, 4, 3) \quad f_1 = [a, b^{-1}, 1, b]$$

For the case of n=3, the extending pattern is as follows.

$$\tau_0 = (1, 7, 2)(3, 9, 4)(5, 8, 6) \quad f_0 = [a^{-1}, a, 1, 1, b, b^{-1}, 1, 1, 1]$$
$$\tau_1 = (1, 2, 8)(3, 4, 7)(5, 6, 9) \quad f_1 = [b, 1, a^{-1}, 1, 1, 1, a, b^{-1}, 1]$$

3 Our Research

The monodromy group $\text{Mon}(\beta)$ contains information about the symmetries of a Belyĭ map β. For any Toroidal Belyĭ map γ,

- There is a surjective group homomorphism $\text{Mon}(\beta \circ \gamma) \twoheadrightarrow \text{Mon}(\beta)$.
- The monodromy group $\text{Mon}(\beta \circ \gamma)$ is contained in the wreath product $\text{Mon}(\gamma) \wr \text{Mon}(\beta)$.

Our project goal was to study how the three groups: $\text{Mon}(\beta)$, $\text{Mon}(\beta \circ \gamma)$, and $\text{Mon}(\gamma) \wr \text{Mon}(\beta)$ compare as we vary over Dynamical Belyĭ maps β and now **Toroidal Belyĭ** maps γ. Our motivating question was: when is $\text{Mon}(\beta \circ \gamma)$ *equal* to $\text{Mon}(\gamma) \wr \text{Mon}(\beta)$?

3.1 Sagemath Code

Using Sagemath, we developed a function which would input a Belyĭ pair and compute helpful information regarding the monodromy group. Below is a description of the code's function:

0. Inputs a Belyĭ pair (f, β) where β is written b in our code.
1. Solve for a list of N points (x, y) such that $f = 0$ and $b = z_0 = \frac{1}{2}$.
2. Solve the first order IVP:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = 2\pi\sqrt{-1}\frac{\beta(x,y) - e}{(\partial\beta/\partial x)(\partial f/\partial y) - (\partial\beta/\partial y)(\partial f/\partial x)} \begin{bmatrix} +\frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial x} \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = P_a$$

   We use Euler's method to do this in Sage.

3. Form a list of endpoints by carrying out step 2 for $a = 1, 2, \ldots, N$ on the interval $0 \leq t \leq 1$ and selecting the endpoint of each path. Do this twice to create 2 lists, one for $e = 0$ and one for $e = 1$.
4. Compare the list of endpoints computed to the list of N points, and take the point P_a from step 1 which is closest to that endpoint. This will help us avoid small rounding errors.
5. Calculate σ_0 and σ_1 by permuting the points in the updated list and returning these permutations as cycles. Find σ_∞ by computing ${\sigma_1}^{-1}{\sigma_0}^{-1}$. This yields the **monodromy triple**.
6. Compute the **monodromy group** of the Belyĭ pair by defining G as the symmetric group of order N and the monodromy group H as the subgroup of G generated by σ_0 and σ_1.

7. Determine isomorphism. Define M as the monodromy group for the Belyĭ pair (f, b) and C as the monodromy group for the Belyĭ pair (f, b^n). Check if $|C| = m^n n$ (the order of the wreath product.)

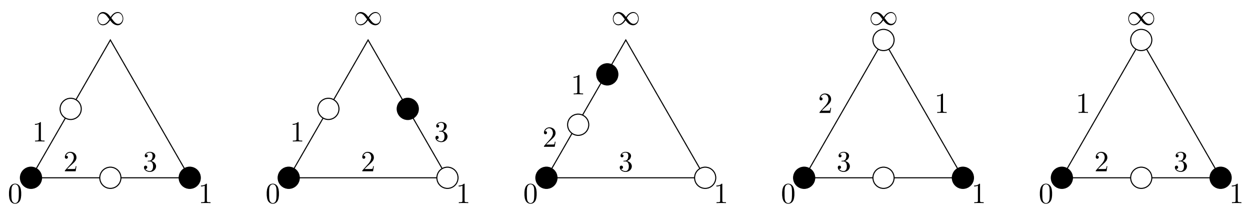
Using this code, we were able to rapidly compute many examples and more easily determine patterns for which the monodromy groups and wreath products were equal. In particular, our code helped us determine for which values of n was this true, and what conditions did the group have to meet (i.e. abelian, cyclic, etc.) in order to satisfy our conjecture.

Example: For the Belyĭ pair $(y^2-(x^3-432), \frac{6x}{y+36})$ where $f = y^2-(x^3-432)$ and $\beta = \frac{6x}{y+36}$, our code returned the monodromy triple $\langle id, (132), (123)\rangle$ and thus the monodromy group is generated by id and (132). If we take $\beta = z^n$, we can run our code again for β^2 and β^3 such that $n = 2$ and $n = 3$, respectively. When we check for isomorphism between the monodromy group and the wreath product, we find that for $n = 2$ their sizes are 9 and 81, meaning that they are not isomorphic and thus that the wreath product is not equal to the monodromy group. In contrast, for the $n = 3$ case both have orders of 18, showing that the monodromy group is equal to the wreath product in this case.

3.2 Extending Pattern Examples

i	$\beta(t)$	Extending Pattern	Generators
1	$-\frac{27}{4}(t^3-t^2)$	$\tau_0 = (12)$ $f_0 = [a,1,b]$ $\tau_1 = (23)$ $f_1 = [1,1,1]$	$[a^{-2}, b^{-1}, b^{-1}], [1,1,1], [b^{-1}, a^{-2}, b^{-1}]$, $[ab^{-1}a^{-1}, b^{-1}, ba^{-2}b^{-1}]$, $[a^{-1}, ab^{-1}, ba^{-1}b^{-1}]$
2	$-2t^3+3t^2$	$\tau_0 = (12)$ $f_0 = [a,1,1]$ $\tau_1 = (23)$ $f_1 = [1,b,1]$	$[a^{-1}, a^{-1}, 1], [1, b^{-1}, b^{-1}], [ab^{-1}a^{-1}, 1, b^{-1}]$, $[a^{-1}, 1, a^{-1}], [1, a^{-1}, a^{-1}]$, $[1, ba^{-1}, 1], [ab^{-1}a^{-1}, a^{-1}, 1]$
3	$\frac{t^3+3t^2}{4}$	$\tau_0 = (23)$ $f_0 = [1,a,1]$ $\tau_1 = (12)$ $f_1 = [1,1,b]$	$[1, a^{-1}, a^{-1}], [1,1,b^{-2}], [a, ab^{-2}a^{-1}, 1]$, $[a^{-1}, 1, ba^{-1}b^{-1}], [a^{-1}, aba^{-1}b^{-1}a^{-1}, 1]$, $[aba^{-1}, b^{-1}a^{-1}, b], [ab^{-1}a^{-1}, b^{-1}a^{-1}, b]$
4	$\frac{27t^2(t-1)}{(3t-1)^3}$	$\tau_0 = (23)$ $f_0 = [b,a,1]$ $\tau_1 = (12)$ $f_1 = [b^{-1}a^{-1},1,1]$	$[b^{-2}, a^{-1}, a^{-1}], [ab, ab, 1], [ba, 1, ab]$, $[b^{-1}a^{-1}b, b^{-2}, a^{-1}], [a^{-1}, a^{-1}, b^{-2}]$, $[b^{-1}, a^{-2}, b^{-1}], [b^{-1}, ba^{-1}, a]$
5	$\frac{t^2(t-1)}{(t-\frac{4}{3})^3}$	$\tau_0 = (12)$ $f_0 = [a,1,b]$ $\tau_1 = (23)$ $f_1 = [b^{-1}a^{-1},1,1]$	$[a^{-1}, a^{-1}, b^{-2}], [b^{-1}a^{-1}b, b^{-2}, a^{-1}]$, $[abab, 1, 1], [1, abab, 1]$, $[ab^{-2}a^{-1}, b^{-1}a^{-1}b, ba^{-1}b^{-1}]$, $[b^{-2}a^{-1}, b^2, b^{-1}a^{-1}b^{-1}], [b^{-2}a^{-1}, b^2, a]$

Melanie Wood provides 5 Belyĭ extending maps labelled β_i for i on range 1 to 5. The extending patterns of those maps and their dessins are summarized in the table above. These extending pattern values became useful when generalizing our proof in the following section. Our Sage code yielded a list of generators for each example, and we were able to compute the minimal generating set for each β in order to make some observations about the monodromy of each map.



β_1 β_2 β_3 β_4 β_5

3.3 Results

We first consider the case where $\mathrm{Mon}(\gamma)$ is abelian and $\beta(z) = z^n$. Before computing $\mathrm{Mon}(\beta\gamma)$ we require a few basic facts about β.

Proposition 1. *The monodromy group, $Mon(\beta) = \langle \tau_0, \tau_1 \rangle$ where $\tau_0 = (1, 2, \ldots, n)$, $\tau_1 = id$, and $Mon(\beta) = C_n$. We also have $f_0 = (1, \ldots, 1, a, 1, \ldots, 1)$ (where a is in entry $\lfloor \frac{n}{2} \rfloor$ + 1 of f_0) and $f_1 = (b, 1, \ldots, 1)$.*

Proof. We begin by describing the Dessin d'Enfant for $\beta(z) = z^n$. We have the sets of vertices,

$$B = \beta^{-1}(\{0\}) = \{0\}$$
$$W = \beta^{-1}(\{1\}) = \{e^{2\pi ik/n} : 0 \leq k < n\}$$

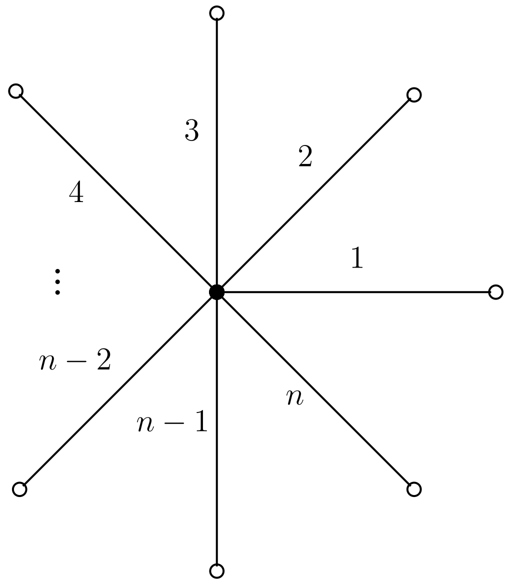
along with the set of edges $E_\beta = \beta^{-1}([0, 1])$. Fix the labeling on E_β where the edge connecting 0 and $e^{2\pi ik/n}$ is labeled $k+1$. Then τ_0 sends every edge k to the edge $k+1$ mod k so that $\tau_0 = (1, 2, \ldots, n)$. For each white vertex, τ_1 sends the edge k back to itself so that $\tau_1 = id$. It follows that

$$\mathrm{Mon}(\beta) = \langle (1, 2, \ldots, n) \rangle \cong C_n.$$

When calculating f_0, the only edge of interest is $\lfloor \frac{n}{2} \rfloor + 1$ because the loop around it crosses the real axis in $(-\infty, 0]$. Since the loop around this edge crosses from $\overline{\mathbb{H}^+}$ to $\mathbb{H}^-$ it contributes an a to the $\lfloor \frac{n}{2} \rfloor + 1$ position of f_0. The loop for each other edge crosses the real axis on $[0, 1]$ or does not cross at all. When calculating f_1, the only edges of interest are 1 and $\frac{n}{2} + 1$ (if it exists). The loop around edge 1 crosses the real axis in $[1, \infty)$ and travels from $\mathbb{H}^-$ to $\overline{\mathbb{H}^+}$. Thus, edge 1 contributes b to the first entry of f_1. The loop around edge $\frac{n}{2} + 1$ crosses the real axis twice in $(-\infty, 0]$ and travels from $\overline{\mathbb{H}^+}$ to $\mathbb{H}^-$ and back to $\overline{\mathbb{H}^+}$. Thus, edge $\frac{n}{2} + 1$ contributes $bb^{-1} = 1$ to position $\frac{n}{2} + 1$ of f_1. It follows that

$$f_0 = (1, \ldots, 1, a, 1, \ldots, 1)$$
$$f_1 = (b, 1, \ldots, 1).$$

□



Dessin d'Enfant for $\beta(z) = z^n$.

Recall that Theorem 4.18 from Jacob Bond's thesis allows us to compute $\mathrm{Mon}(\beta\gamma)$ by instead computing $\rho_{\gamma^*}(A)$, where A is defined as $\varphi(\mathrm{Ker}\rho_\beta)$. We begin by finding the generators of $\mathrm{Ker}\rho_\beta$.

Proposition 2. *$Ker(\rho_\beta) = \langle b, a^n, a^i b a^{-i}\rangle$ for $i \in \{\pm 1, \ldots, \pm\lfloor \frac{n}{2} \rfloor\}$.*

Proof. Recall from topology that $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) = F_2$ so that $\rho_\beta : F_2 \to \mathrm{Mon}(\beta)$. Here F_2 denotes the free group on two generators $\langle a, b\rangle$ where a represents the class of loops around 0, and b represents the class of loops around 1. Proposition 1 then tells us that

$$\rho_\beta(a) = \tau_0 = (1, 2, \ldots, n)$$
$$\rho_\beta(b) = \tau_1 = id.$$

We can then determine $\rho_\beta(b) = \rho_\beta(a^n) = \rho_\beta(a^i b a^{-i}) = id$ for $i \in \{\pm 1, \ldots, \pm\lfloor \frac{n}{2} \rfloor\}$. It follows that $\rho_\beta\left(\langle b, a^n, a^i b a^{-i}\rangle\right) = \{id\}$. Since $\langle b\rangle \subseteq \langle b, a^n, a^i b a^{-i}\rangle$, we have

$$b \cdot \langle b, a^n, a^i b a^{-i}\rangle \cdot b^{-1} \subseteq \langle b, a^n, a^i b a^{-i}\rangle.$$

Conjugation by a is slightly more complex. We have

$$a \cdot a^n \cdot a^{-1} = a^n \in \langle b, a^n, a^i b a^{-i}\rangle$$
$$a \cdot b \cdot a^{-1} = aba^{-1} \in \langle b, a^n, a^i b a^{-i}\rangle.$$

Consider $a \cdot a^i b a^{-i} \cdot a^{-1} = a^{i+1} b a^{-(i+1)}$. If $i = \lfloor \frac{n}{2} \rfloor$, then $a^{\lfloor \frac{n}{2} \rfloor + 1} b a^{-(\lfloor \frac{n}{2} \rfloor + 1)}$ is not explicitly a generator. However,

$$a^{\lfloor \frac{n}{2} \rfloor + 1} b a^{-(\lfloor \frac{n}{2} \rfloor + 1)} = a^n \cdot \left(a^{-\lfloor \frac{n}{2} \rfloor + 1} b a^{\lfloor \frac{n}{2} \rfloor - 1}\right) \cdot a^{-n} \in \langle b, a^n, a^i b a^{-i}\rangle.$$

Thus, $\langle b, a^n, a^iba^{-i}\rangle \trianglelefteq F_2$. Notice that

$$F_2/\langle b, a^n, a^iba^{-i}\rangle = \{\overline{id}, \overline{a}, \dots, \overline{a^{n-1}}\} \cong C_n.$$

Since $\rho_\beta(\langle b, a^n, a^iba^{-i}\rangle) = \{id\}$ and $F_2/\langle b, a^n, a^iba^{-i}\rangle \cong \mathrm{Mon}(\beta)$ it follows that

$$\mathrm{Ker}(\rho_\beta) = \langle b, a^n, a^iba^{-i}\rangle.$$

□

Now that we have the generators of $\mathrm{Ker}\rho_\beta$, we can determine where each is sent by φ using the semi-direct product group law.

Lemma 3.1. *Given the homomorphism φ such that*

$$\varphi(a) = [f_0, \tau_0]$$
$$\varphi(b) = [f_1, \tau_1]$$

we obtain the following relations:

$$\varphi(b) = [(b, 1, \dots, 1); id] \tag{1}$$
$$\varphi(a^n) = [(a, \dots, a); id] \tag{2}$$
$$\varphi(a^iba^{-i}) = [(1, \dots, 1, d, 1, \dots, 1); id]. \tag{3}$$

Here, d is in the i^{th} position of $\varphi(a^iba^{-i})$ and

$$d = \begin{cases} aba^{-1} & \text{if } |i| = \lfloor \frac{n}{2} \rfloor \\ b & \text{otherwise} \end{cases}$$

Proof. Note from Proposition 1 that

$$\varphi(a) = [(1, \dots, 1, a, 1, \dots, 1); (1, 2, \dots, n)]$$
$$\varphi(b) = [(b, 1, \dots, 1); id].$$

This proves relation (1) immediately. Before proving (2) and (3), we derive a general form for $\varphi(a)^k$ where $1 \leq k \leq n$. We do so by induction, where our base case $k = 1$ is given above. Assume for some $1 \leq \ell \leq n$ that

$$\varphi(a)^\ell = [(1, \dots, 1, a, \dots, a, 1, \dots, 1); \tau_0^\ell]$$

where a appears in positions $\lfloor \frac{n}{2} \rfloor + 1$ through $\lfloor \frac{n}{2} \rfloor + \ell$ modulo n. Then

$$\begin{aligned} \varphi(a)^{\ell+1} &= \varphi(a) \cdot \varphi(a)^\ell \\ &= [(1, \dots, a, \dots, 1); \tau_0] \cdot [(1, \dots, a, \dots, a, \dots, 1); \tau_0^\ell] \\ &= [(1, \dots, a, \dots, 1) \cdot \tau_0(1, \dots, a, \dots, a, \dots, 1); \tau_0^{\ell+1}]. \end{aligned}$$

Note that $\tau_0(1,\dots,a,\dots,a,\dots,1)$ has a in positions $\lfloor \frac{n}{2} \rfloor + 2$ through $\lfloor \frac{n}{2} \rfloor + \ell + 1$. Since $\varphi(a)$ has a single a in position $\lfloor \frac{n}{2} \rfloor + 1$ modulo n, it follows that

$$\varphi(a)^{\ell+1} = [(1,\dots,a,\dots,a,\dots,1); \tau_0^{\ell+1}]$$

where a is in positions $\lfloor \frac{n}{2} \rfloor + 1$ through $\lfloor \frac{n}{2} \rfloor + \ell + 1$. Thus, by induction, for any $1 \le i \le n$ we have

$$\varphi(a)^i = [(1,\dots,a,\dots,a,\dots,1); \tau_0^i]$$

where a is in positions $\lfloor \frac{n}{2} \rfloor + 1$ through $\lfloor \frac{n}{2} \rfloor + i$ modulo n. Relation (2) follows from the $i = n$ case. A similar proof shows the following for $-n \le -i \le -1$:

$$\varphi(a)^{-i} = [(1,\dots,a^{-1},\dots,a^{-1},\dots,1); \tau_0^{-i}]$$

where a^{-1} appears in positions $\lfloor \frac{n}{2} \rfloor - i$ through $\lfloor \frac{n}{2} \rfloor$ modulo n. Now, consider for some $1 \le i \le n$

$$\begin{aligned}\varphi(ba^{-i}) &= [(b,1,\dots,1); id] \cdot [(1,\dots,a^{-1},\dots,a^{-1},\dots,1); \tau_0^{-i}]\\ &= [(b,1,\dots,1) \cdot (1,\dots,a^{-1},\dots,a^{-1},\dots,1); \tau_0^{-1}]\\ &= [(c,\dots,a^{-1},\dots,a^{-1},\dots,1); \tau_0^{-i}]\end{aligned}$$

where c denotes ba^{-1} when $i = \lfloor \frac{n}{2} \rfloor$ and b otherwise. For that same i consider the following:

$$\begin{aligned}\varphi(a^i ba^{-i}) &= \varphi(a) \cdot \varphi(ba^{-i})\\ &= [(1,\dots,a,\dots,a,\dots,1); \tau_0^i] \cdot [(c,\dots,a^{-1},\dots,a^{-1},\dots,1); \tau_0^{-i}]\\ &= [(1,\dots,a,\dots,a,\dots,1) \cdot \tau_0^i(c,\dots,a^{-1},\dots,a^{-1},\dots,1); id].\end{aligned}$$

Note that $\tau_0^i(c,\dots,a^{-1},\dots,a^{-1},\dots,1)$ has c in position i and a^{-1} in positions $\lfloor \frac{n}{2} \rfloor + 1$ through $\lfloor \frac{n}{2} \rfloor + i$. This gives relation (3)

$$\begin{aligned}\varphi(a^i ba^{-i}) &= [(1,\dots,d,aa^{-1},\dots,aa^{-1},\dots,1); id]\\ &= [(1,\dots,d,\dots,1); id].\end{aligned}$$

where $d = aba^{-1}$ when $|i| = \lfloor \frac{n}{2} \rfloor$ and b otherwise. $\square$

Using Lemma 1, we can finally determine a general form for $\rho_{\gamma^*}(A)$ when $\mathrm{Mon}(\gamma)$ is abelian and $\beta(z) = z^n$.

Proposition 3. *$\rho_{\gamma^*}(A) = \langle (b_\gamma,1,\dots,1),\dots,(1,\dots,1,b_\gamma),(a_\gamma,\dots,a_\gamma)\rangle$ where b_γ appears in each of n positions.*

Proof. Recall from Lemma 1 that $\varphi(a^i ba^{-i}) = [(1,\dots,1,d,1,\dots,1); id]$ where d is in the i^{th} position. As i ranges over $\{0, \pm 1, \dots, \pm\lfloor \frac{n}{2} \rfloor\}$ we get n distinct generators of A:

$$\{[(b,1,\dots,1); id], [(1,d,1,\dots,1); id], \dots, [(1,\dots,1,d); id]\}.$$

Combined with the last generator, $\varphi(a^n)$ it follows that

$$A = \langle [(b, 1, \ldots, 1); id], [(1, d, 1, \ldots, 1); id], \ldots, [(1, \ldots, 1, d); id], [(a, \ldots, a); id] \rangle.$$

Note that under the assumption $\mathrm{Mon}(\gamma)$ is abelian, $a_\gamma b_\gamma a_\gamma^{-1} = b_\gamma$ so that $\rho_{\gamma^*}(d) = b_\gamma$. Thus,

$$\rho_{\gamma^*}(A) = \langle (b_\gamma, 1, \ldots, 1), \ldots, (1, \ldots, 1, b_\gamma), (a_\gamma, \ldots, a_\gamma) \rangle.$$

□

In our final step, we use Proposition 3 to determine the exact conditions under which $\mathrm{Mon}(\beta\gamma) \cong \mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$.

Theorem 3.2. *Let $Mon(\gamma) = \langle a_\gamma, b_\gamma \rangle$ be abelian and $\beta(z) = z^n$ for some $n > 1$. Then $Mon(\beta\gamma) \cong Mon(\gamma) \wr Mon(\beta)$ if and only if $Mon(\gamma) = \langle b_\gamma \rangle$.*

Proof. We begin with the forward direction. Assume $\mathrm{Mon}(\beta\gamma) \cong \mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$. Theorem 4.18 from Jacob Bond's thesis tells us this is equivalent to saying $\rho_{\gamma^*}(A) \cong (\mathrm{Mon}(\gamma))^n$. Note that $(a_\gamma, 1, \ldots, 1) \in \rho_{\gamma^*}(A)$. So we can fix $k_1, k_2, \ell \in \mathbb{Z}$ such that

$$\begin{aligned} a_\gamma &= b_\gamma^{k_1} \cdot a_\gamma^\ell \\ 1 &= b_\gamma^{k_2} \cdot a_\gamma^\ell. \end{aligned}$$

We can solve to get $a_\gamma^\ell = b_\gamma^{-k_2}$. Plugging this in gives $a_\gamma = b_\gamma^{k_1 - k_2}$. Thus, $a_\gamma \in \langle b_\gamma \rangle$. Since $\mathrm{Mon}(\gamma) = \langle a_\gamma, b_\gamma \rangle$ and $\langle a_\gamma \rangle \subseteq \langle b_\gamma \rangle$, it follows that $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$.
Now, the reverse direction. Assume $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$. Note that

$$\langle (b_\gamma, 1, \ldots, 1), \ldots, (1, \ldots, 1, b_\gamma) \rangle \leq \rho_{\gamma^*}(A).$$

We can then recognize the following isomorphism:

$$\langle (b_\gamma, 1, \ldots, 1), \ldots, (1, \ldots, 1, b_\gamma) \rangle \cong \langle b_\gamma \rangle^n \cong (\mathrm{Mon}(\gamma))^n.$$

It follows that $(\mathrm{Mon}(\gamma)) \cong \rho_{\gamma^*}(A)$. Using Theorem 4.18 from Jacob Bond's thesis we can conclude that $\mathrm{Mon}(\beta\gamma) \cong \mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$. □

We can prove analogous results for other dynamical Belyĭ maps.

Proposition 4. *Suppose γ toroidal Belyĭ map with $Mon(\gamma)$ abelian. Let β_i denote the dynamical Belyĭ map of the same name given in Section 3.2. Then we have the following sufficient conditions for when $Mon(\beta_i\gamma) \cong Mon(\gamma) \wr Mon(\beta_i)$:*

- β_1: *$Mon(\gamma) = \langle a_\gamma^2 \rangle$ or $a_\gamma = 1$ (so that $Mon(\gamma) = \langle b_\gamma \rangle$)*
- β_2: *$Mon(\gamma) = \langle a_\gamma^2 \rangle$ or $Mon(\gamma) = \langle b_\gamma^2 \rangle$*
- β_3: *$Mon(\gamma) = \langle a_\gamma^2 \rangle$ or $Mon(\gamma) = \langle b_\gamma^2 \rangle$*
- β_4: *$Mon(\gamma) = \langle c_\gamma^2 \rangle$*
- β_5: *$Mon(\gamma) = \langle c_\gamma^2 \rangle$*

3.4 Further Research

Our work is in no way comprehensive. Areas of future study which may yield interesting results would include an examination of monodromy groups which are non-abelian, since our proof exclusively explored the abelian case. Another variable which could be considered is in the makeup of our Belyĭ map composition. Our map was toroidal, but there are other cases which could include surfaces with genus > 1, and even when limiting our case to a genus ≤ 1, there were discrepancies between our Belyĭ composition and Wood's or Zeytin's.

References

[1] J.A. Bond. *On the Computation and Composition of Belyĭ Maps and Dessins D'enfants*. 2018. http://gateway.proquest.com.ezproxy.library.wisc.edu/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:10809531.

[2] M. Mabe, R.A. Moy, J. Schmurr, and J. Varlack. Monodromy Groups of Dessins d'Enfant on Rational Triangular Billiard Surfaces, 2021. https://arxiv.org/abs/2106.15588v1.

[3] M.M. Wood. Belyi-extending maps and the Galois action on dessins d'enfants. *Publications of the Research Institute for Mathematical Sciences*, 42(3):721–738, 2006. http://projecteuclid.org.ezproxy.library.wisc.edu/euclid.prims/1166642157.

[4] A. Zeytin. Belyĭ Lattès Maps, 2016. https://arxiv.org/abs/1011.5644.

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